

# ON LOW TEMPERATURE KINETIC THEORY; SPIN DIFFUSION, BOSE EINSTEIN CONDENSATES, ANYONS

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## Abstract

The paper considers some typical problems for kinetic models evolving through pair-collisions at temperatures not far from absolute zero, which illustrate specifically quantum behaviours. Based on these - mostly recent - examples, a number of differences between quantum and classical Boltzmann theory is then discussed in more general terms.

## 0 Introduction.

Let us start by recalling some basic facts about **the classical Boltzmann case**. Consider a parcel of gas evolving as

$$\frac{dx}{dt} = v, \frac{dv}{dt} = F.$$

That implies the evolution of the gas density  $f$ ;

$$D_t f(x(t), v(t), t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial v} \frac{dv}{dt} = \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + F \frac{\partial f}{\partial v},$$

which gives the classical BE when driven by a collision term  $Q(f)$ ,

$$(D_t f(x(t), v(t), t) =) \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + F \frac{\partial f}{\partial v} = Q(f).$$

For two particles having pre-collisional velocities  $v, v_*$ , denote the velocities after collision by  $v', v'_*$ , and write  $f_* = f(v_*)$ ,  $f' = f(v')$ ,  $f'_* = f(v'_*)$ . In this notation the classical Boltzmann collision operator becomes  $Q(f) = \int B(f'f'_* - ff_*) dv_* d\omega$ . The kernel  $B$  is typically of the type  $b(\omega)|v - v_*|^s$  with  $-3 < s \leq 1$ . The collision term is linearly proportional to the density of each of the two participating molecules. Mass and kinetic energy are each conserved, and the monotone in time entropy  $\int dv dx f \log f(t)$  prevents strong concentrations to form. In the classical case *all collisions* respecting the conservation laws are permitted,

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**The quantum case** is different. Close to absolute zero, permitted energy levels are often discrete, implying fewer collisions. There may exist a condensate, where an excitation may interact only with those modes of its zero-point motion that will not give away energy. In the quantum regime, also the thermal de Broglie wave length may become much larger than the typical inter-particle distance, a situation totally different from classical kinetic theory. The scattering is wave-like and two-body quantum statistics gives a marked imprint on the individual collision process, e.g. the Pauli principle for fermions.

The rich phenomenology of low temperature gases corresponds to a similarly rich mathematical structure for its models, including the kinetic ones. The quantum kinetic collisions and equations are much more varied and fine-tuned than in the classical case. This paper will consider typical quantum kinetic models evolving through pair collisions, in order to illustrate the differences between the classical and quantum cases;

- 1) The Nordheim Uehling Uhlenbeck model,
- 2) Spin in the fermionic case,
- 3) Low temperature bosons and a condensate,
- 4) A kinetic anyon equation.

These models are obtained by physics arguments from wave mechanics in the Heisenberg setting and from quantum field theory. In some cases the models have been validated in a formal mathematical sense (see [BCEP], [S1]). Based on the examples 1-4, the final section will elaborate on some

- 5) Differences between quantum and classical Boltzmann theory.

## 1 The Nordheim Uehling Uhlenbeck model.

Around 1930 the first quantum kinetic equation for boson and fermion gases was proposed independently by Nordheim ([N]) and by Uehling-Uhlenbeck ([UU]). The equation is of particular interest in a neighbourhood of and below the transition temperature  $T_c$ , where a condensate first appears. The NUU equation is *semiclassical*; the transport left hand side of this evolution equation is classical, only the collisional right hand side is directly influenced by the quantum effects. The term quantum Boltzmann equation usually refers to a semiclassical equation, even though true quantum kinetic models are sometimes considered.

The collision operator for the NUU equation is

$$Q_{NUU}(f)(p) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} B \delta(p + p_* - (p' + p'_*)) \delta(E(p) + E(p_*) - (E(p') + E(p'_*))) \\ \left( f' f'_* (1 + \epsilon f)(1 + \epsilon f_*) - f f_* (1 + \epsilon f')(1 + \epsilon f'_*) \right) dp_* dp' dp'_*, \quad (1.1)$$

with  $\epsilon = 1$  for bosons,  $\epsilon = -1$  for fermions, and  $\epsilon = 0$  gives the classical case. Here the quartic terms vanish. A discussion of the validation can be found in [BCEP].

For  $\epsilon = \pm 1$  the entropy is

$$\int \left( f \log f - \frac{1}{\epsilon} (1 + \epsilon f) \log(1 + \epsilon f) \right) dp.$$

Also this entropy is monotone but, contrary to the classical case, gives *no control of concentrations*. In equilibrium  $Q$  is zero, and multiplication with  $\log \frac{f}{1+\epsilon f}$  and integration gives

$$\begin{aligned} 0 = \int Q(f) \log \frac{f}{1+\epsilon f} dv &= \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} B \delta(p + p_* - (p' + p'_*)) \delta(E(p) + E(p_*) - (E(p') + E(p'_*))) \\ &\quad (1 + \epsilon f)(1 + \epsilon f_*)(1 + \epsilon f')(1 + \epsilon f'_*) \left( \frac{f'}{1 + \epsilon f'} \frac{f'_*}{1 + \epsilon f'_*} - \frac{f}{1 + \epsilon f} \frac{f_*}{1 + \epsilon f_*} \right) \\ &\quad \left( \log \frac{f}{1 + \epsilon f} \log \frac{f_*}{1 + \epsilon f_*} - \log \frac{f'}{1 + \epsilon f'} \log \frac{f'_*}{1 + \epsilon f'_*} \right). \end{aligned}$$

The two last factors in the integral have the same sign, which implies

$$\frac{f'}{1 + \epsilon f'} \frac{f'_*}{1 + \epsilon f'_*} \equiv \frac{f}{1 + \epsilon f} \frac{f_*}{1 + \epsilon f_*}.$$

As in the classical case, we conclude that  $\frac{f}{1+\epsilon f} = M$ , a Maxwellian, hence  $f = \frac{M}{1-\epsilon M}$ , which for  $\epsilon = \pm 1$  is the *Planckian* equilibrium function.

For fermions ( $\epsilon = -1$ ) the concentrations are bounded by one, because of the factor  $(1 - f)$  which preserves positivity together with  $f$ . This is stronger than the entropy controlled concentrations from the classical case. There are general existence results for the fermion case in an  $L^1 \cap L^\infty$ -setting due to J. Dolbeault [D] and P.L. Lions [PLL]. The boson case is more intricate. So far there are only partial existence results - for the space-homogeneous, isotropic setting ([Lu1], [Lu2], [EMV]), and for some space-dependent settings close to equilibrium ([R]).

Mathematical results have also been obtained concerning existence, uniqueness and asymptotic behaviour ([EM]) for the related **Boltzmann-Compton equation**

$$k^2 \frac{\partial f}{\partial t} = \int_0^\infty (f'(1 + f)B(k', k; \theta) - f(1 + f')B(k, k'; \theta)) dk' \quad (1.2)$$

with  $f$  photon density,  $k$  energy,  $\theta$  temperature, and with the detailed balance law  $e^{k/\theta} B(k', k; \theta) = e^{k'/\theta} B(k, k'; \theta)$ . This equation models a space-homogeneous isotropic photon gas in interaction with a low temperature gas of electrons in equilibrium, having a *Maxwellian distribution* of velocities.

## 2 a) Spin in the fermionic case.

The experimental study of spin polarized neutral gases at low temperatures and their kinetic modelling is well established in physics, an early mathematical physics text in the area being [S]. The first experiments concerned very dilute solutions of  $^3\text{He}$  in superfluid  $^4\text{He}$  with - in comparison with classical Boltzmann gases - interesting new properties such as spin waves (see [NTLCL]). The experimentalists later turned to laser-trapped low-temperature gases (see [JR]). The mathematical study of these models, however, is less advanced. To discuss this, we first recall some properties of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With  $[\sigma_i, \sigma_j]$  denoting the commutator  $\sigma_i \sigma_j - \sigma_j \sigma_i$ , the Pauli matrices satisfy

$$[\sigma_1, \sigma_2] = 2i\sigma_3, \quad [\sigma_2, \sigma_3] = 2i\sigma_1, \quad [\sigma_3, \sigma_1] = 2i\sigma_2, \quad \text{and} \quad [\sigma_i, \sigma_i] = 0 \quad \text{for} \quad i = 1, 2, 3,$$

With  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$  the Pauli spin matrix vector, this is equivalent to  $\sigma \times \sigma = 2i\sigma$ . Let  $\mathcal{M}_2(\mathbb{C})$  denote the space of  $2 \times 2$  complex matrices, and  $\mathcal{H}_2(\mathbb{C})$  the subspace of hermitean matrices.  $\mathcal{H}_2(\mathbb{C})$  is linearly isomorphic to  $\mathbb{R}^4$ , if we use the decomposition  $\rho = A_c I + A_s \cdot \sigma$  and identify  $\rho \in \mathcal{H}_2(\mathbb{C})$  with  $(A_c, A_s) \in \mathbb{R}^4$ .

A *dilute spin polarized gas* with spin  $\frac{1}{2}$ , will be modelled by a distribution function matrix  $\rho \in \mathcal{H}_2(\mathbb{C})$  which is the Wigner transform of the one-atom density operator for the system being modelled. The domain of  $\rho(t, x, p)$  is taken as positive time  $t$ ,  $p \in \mathbb{R}^3$ , and for simplicity here with position-space  $x$  periodic in 3d with period one. We shall focus on the following simplified model for the kinetic evolution of  $\rho$ ,

$$D\rho := \frac{\partial}{\partial t} \rho + p \cdot \nabla_x \rho = Q(\rho) \tag{2.1}$$

with (in the Born approximation), the collision integral given by

$$Q(\rho) = \int dp_2 dp'_1 dp'_2 B \delta(p_1 + p_2 - p'_1 - p'_2) \delta(p_1^2 + p_2^2 - p'^2_1 - p'^2_2) \\ \left[ \{[\rho_{1'}, \tilde{\rho}_1]_+ \text{Tr}(\tilde{\rho}_2 \rho_{2'}) - [\tilde{\rho}_{1'}, \rho_1]_+ \text{Tr}(\rho_2 \tilde{\rho}_{2'})\} - \frac{1}{8} \{[[\tilde{\rho}_1, \rho_{1'}]_+, [\tilde{\rho}_2, \rho_{2'}]_+]_+ - [[\rho_1, \tilde{\rho}_{1'}]_+, [\rho_2, \tilde{\rho}_{2'}]_+]_+\} \right].$$

Here  $\tilde{\rho} = I - \rho$ , and  $[\cdot, \cdot]_+$  denotes an anti-commutator. The kernel  $B$  is of hard force type,  $0 \leq B(|p_1 - p_2|, \theta) = |p_1 - p_2|^\beta b(\theta)$  with  $0 \leq \beta \leq 1$ ,  $\frac{b(\theta)}{\cos^3(\theta)} \in L^\infty$ . The number density of particles of any spin component, is given by  $f := \text{Tr}(\rho(t, x, p))$ , and the magnetization of particles is given by the vector  $\bar{\sigma}(t, x, p) := \text{Tr}(\sigma \rho(t, x, p))$ , which implies  $\rho = \frac{1}{2}(fI + \bar{\sigma} \cdot \sigma)$ . The resulting equations for  $f$  and  $\bar{\sigma}$  are

$$Df = Q_n(f, \bar{\sigma}), \tag{2.2}$$

$$D\bar{\sigma} = Q_m(f, \bar{\sigma}), \tag{2.3}$$

where

$$Q_n(f, \bar{\sigma}) = \frac{1}{2} \int dp_2 dp'_1 dp'_2 B \delta(p_1 + p_2 - p'_1 - p'_2) \delta(p_1^2 + p_2^2 - p'^2_1 - p'^2_2) \\ \left( \frac{3}{2} ([f_{1'} - \frac{1}{2}(f_1 f_{1'} + \bar{\sigma}_1 \cdot \bar{\sigma}_{1'})][f_{2'} - \frac{1}{2}(f_2 f_{2'} + \bar{\sigma}_2 \cdot \bar{\sigma}_{2'})] - [f_1 - \frac{1}{2}(f_1 f_{1'} + \bar{\sigma}_1 \cdot \bar{\sigma}_{1'})][f_2 - \frac{1}{2}(f_2 f_{2'} + \bar{\sigma}_2 \cdot \bar{\sigma}_{2'})]) \right. \\ \left. - \frac{1}{2} ([\bar{\sigma}_{1'} - \frac{1}{2}(f_{1'} \bar{\sigma}_1 + f_1 \bar{\sigma}_{1'})] \cdot [\bar{\sigma}_{2'} - \frac{1}{2}(f_{2'} \bar{\sigma}_2 + f_2 \bar{\sigma}_{2'})] - [\bar{\sigma}_1 - \frac{1}{2}(f_1 \bar{\sigma}_1 + f_1 \bar{\sigma}_{1'})][\bar{\sigma}_2 - \frac{1}{2}(f_2 \bar{\sigma}_2 + f_2 \bar{\sigma}_{2'})]) \right), \\ Q_m(f, \bar{\sigma}) = \frac{1}{2} \int dp_2 dp'_1 dp'_2 B \delta(p_1 + p_2 - p'_1 - p'_2) \delta(p_1^2 + p_2^2 - p'^2_1 - p'^2_2) \\ \left( \frac{3}{2} ([f_2 - \frac{1}{2}(f_2 f_{2'} + \bar{\sigma}_2 \cdot \bar{\sigma}_{2'})][\bar{\sigma}_1 - \frac{1}{2}(f_{1'} \bar{\sigma}_1 + f_1 \bar{\sigma}_{1'})] - [f_{2'} - \frac{1}{2}(f_2 f_{2'} + \bar{\sigma}_2 \cdot \bar{\sigma}_{2'})][\bar{\sigma}_{1'} - \frac{1}{2}(f_{1'} \bar{\sigma}_1 + f_1 \bar{\sigma}_{1'})]) \right. \\ \left. - \frac{1}{2} ([f_{1'} - \frac{1}{2}(f_1 f_{1'} + \bar{\sigma}_1 \cdot \bar{\sigma}_{1'})][\bar{\sigma}_{2'} - \frac{1}{2}(f_{2'} \bar{\sigma}_2 + f_2 \bar{\sigma}_{2'})] - [f_1 - \frac{1}{2}(f_1 f_{1'} + \bar{\sigma}_1 \cdot \bar{\sigma}_{1'})][\bar{\sigma}_2 - \frac{1}{2}(f_2 \bar{\sigma}_2 + f_2 \bar{\sigma}_{2'})]) \right).$$

The collision term  $Q_n$  does not change the number density ( $\int Q_n dp = 0$ ), the linear momentum density ( $\int p Q_n dp = 0$ ), and the energy density ( $\int p^2 Q_n dp = 0$ ), and the collision term  $Q_m$  does not change the magnetization density ( $\int Q_m dp = 0$ ).

## 2 b) Existence results.

Consider the initial value problem for the equations (2.2-3). The initial value  $(f_0, \bar{\sigma}_0) \in L^\infty$  is assumed for each ball  $p^2 \leq j^2$  to satisfy  $0 < \eta_j \leq f_0 \leq 2 - \eta_j$ ,  $\min((2 - f_0)^2, f_0^2) \geq \bar{\sigma}_0 \cdot \bar{\sigma}_0 + \eta_j^2$  for some  $\eta_j > 0$ . We first treat the case with a truncation  $B_j$  in the domain of integration for  $Q$ , where  $B_j$  is the restriction of  $B$  to the set  $p_1^2 + p_2^2 \leq j^2$ .

Set

$$\begin{aligned} F(t, x, p) &= f(t, x, p) \quad \text{for } 0 \leq f \leq 2, \quad = 0 \quad \text{for } f < 0, \quad = 2 \quad \text{for } f > 2, \\ \Sigma(t, x, p) &= \bar{\sigma}(t, x, p) \quad \text{when } \min(F^2, (2 - F)^2) \geq \bar{\sigma} \cdot \bar{\sigma}, \\ &\quad \text{else } \Sigma(t, x, p) = \frac{\min(F, 2 - F)\bar{\sigma}}{\sqrt{\bar{\sigma} \cdot \bar{\sigma}}}(t, x, p). \end{aligned}$$

The system  $Df = Q_n(F, \Sigma)$ ,  $D\bar{\sigma} = Q_m(F, \Sigma)$  with the initial value  $(f_0, \bar{\sigma}_0)$  under the truncation  $B_j$  can be solved by a contraction argument.

**Lemma 2.1** [A2] *The initial value problem with initial values  $(f_0, \sigma_0)$  for the truncated problem  $Df = Q_n(F, \Sigma)$ ,  $D\bar{\sigma} = Q_m(F, \Sigma)$  with truncation  $B_j$ , has a unique local solution in  $L^\infty$ .*

It remains to prove that  $F = f$  and  $\Sigma = \bar{\sigma}$ . This holds by continuity on a (short,  $\eta$ - and  $j$ -dependent) time interval  $0 \leq t < t_j$ , using the boundedness of  $T_n$  and  $T_m$ . We give the proof for general  $t$ , when there is spin only in the  $\sigma_3$  direction, and refer to the paper [A2] for the general case. With  $\bar{\sigma} = (0, 0, s)$ , we can replace  $\bar{\sigma}$  with  $s$  in  $Q_n$  and  $Q_m$ , and the equation for  $\bar{\sigma}$  by the corresponding one for  $s$ . Set  $c_1 = 18 \max_{p_1} \int dp_2 dp'_1 dp'_2 B_n \delta(p_1 + p_2 - p'_1 - p'_2) \delta(p_1^2 + p_2^2 - p'^2_1 - p'^2_2)$ . It holds

$$\begin{aligned} -c_1(f_1^2 - s_1^2) &\leq \int dp_2 dp'_1 dp'_2 B \delta(p_1 + p_2 - p'_1 - p'_2) \delta(p_1^2 + p_2^2 - p'^2_1 - p'^2_2) \\ &\quad \frac{1}{2} \left( - (f_1 - s_1) \left( [(f_1 + s_1)(1 - \frac{1}{2}(f_{1'} + s_{1'}))] [(f_2 - s_2)(1 - \frac{1}{2}(f_{2'} - s_{2'}))] \right) \right. \\ &\quad \left. - (f_1 + s_1) \left( [(f_1 - s_1)(1 - \frac{1}{2}(f_{1'} - s_{1'}))] [(f_2 + s_2)(1 - \frac{1}{2}(f_{2'} + s_{2'}))] \right) \right) \\ &\quad + \frac{1}{4} \left( - (f_1 - s_1) \left( [(f_1 + s_1)(1 - \frac{1}{2}(f_{1'} + s_{1'}))] [(f_2 + s_2)(1 - \frac{1}{2}(f_{2'} + s_{2'}))] \right) \right. \\ &\quad \left. - (f_1 + s_1) \left( [(f_1 - s_1)(1 - \frac{1}{2}(f_{1'} - s_{1'}))] [(f_2 - s_2)(1 - \frac{1}{2}(f_{2'} - s_{2'}))] \right) \right) \\ &\leq (f_1 - s_1)(Q_n + Q_m)(f, s) + (f_1 + s_1)(Q_n - Q_m)(f, s) = D(f_1^2 - s_1^2). \end{aligned} \tag{2.4}$$

This implies

$$0 < (f_0^2 - s_0^2)(x, p) e^{-c_1 t} \leq (f^2 - s^2)^\#(t, x, p), \tag{2.5}$$

where  $(f, s)$  is the contraction solution with initial value  $(f_0, \bar{\sigma}_0)$ . By continuity  $f \pm s > 0$  for  $0 \leq t \leq t_j$ . Analogously, starting from the equation for  $(1 - \frac{1}{2}(f_1 \pm s_1))$  instead of  $f_1 \pm s_1$ , we get by uniqueness the same solution, together with the estimate

$$0 < ((2 - f_0)^2 - s_0^2)(x, p) e^{-c_1 t} \leq ((2 - f)^2 - s^2)^\#(t, x, p), \tag{2.6}$$

which by continuity holds for  $0 \leq t \leq t_j$  together with  $2 - f \pm s > 0$  for  $0 \leq t \leq t_j$ . And so uniformly in  $(x, p)$ ,  $|s| < \min(f, 2 - f) \leq 1$  for  $0 \leq t \leq t_j$ . By iteration, existence and uniqueness follow for  $t > 0$ . We conclude that  $\rho = \frac{1}{2}(fI + \bar{\sigma}\sigma)$ , solves the truncated initial value problem for (2.1) globally in time.

We can remove the truncation using a variant of the particular limit procedure from [PLL] for  $\rho_j$ , when  $j \rightarrow \infty$ .

**Theorem 2.2** [A2] *Suppose that  $(f_0, \bar{\sigma}_0) \in L^\infty \cap L^1([0, 1]^3 \times \mathbb{R}^3)$ , and for  $p^2 \leq j^2$ ,  $j \in \mathbb{N}$ , that  $0 < \eta_j \leq f_0 \leq 2 - \eta_j$  and  $\bar{\sigma}_0^2 + \eta_j^2 \leq \min(f_0^2, (2 - f_0)^2)$  for some  $\eta_j > 0$ . Then the system (2.2-3) with initial value  $(f_0, \bar{\sigma}_0)$ , has a bounded integrable solution for  $t > 0$  with  $0 < f < 2$  and  $\bar{\sigma}^2 < \min(f^2, (2 - f)^2)$ .*

For details see [A2]. Other problems of considerable physical interest are (cf [JM]) the relaxation times for spin-diffusion, the time asymptotic behaviour in general, and the influence of more involved transport terms in (2.2-3) such as the physicists' version of the problem, namely

$$\begin{aligned} \frac{\partial f}{\partial t} + \nabla_p \epsilon_p \cdot \nabla_r f_p - \nabla_r \epsilon_p \cdot \nabla_p f_p + \sum_{i=xyz} \left[ \frac{\partial h_p}{\partial p_i} \cdot \frac{\partial \bar{\sigma}_p}{\partial r_i} - \frac{\partial h_p}{\partial r_i} \cdot \frac{\partial \bar{\sigma}_p}{\partial p_i} \right] &= Q_n \\ \frac{\partial \bar{\sigma}_p}{\partial t} + \sum_i \left[ \frac{\partial \epsilon_p}{\partial p_i} \frac{\partial \bar{\sigma}_p}{\partial r_i} - \frac{\partial \epsilon_p}{\partial r_i} \frac{\partial \bar{\sigma}_p}{\partial p_i} + \frac{\partial f_p}{\partial r_i} \frac{\partial h_p}{\partial p_i} - \frac{\partial f_p}{\partial p_i} \frac{\partial h_p}{\partial r_i} \right] - 2(h_p \times \bar{\sigma}_p) &= Q_m. \end{aligned}$$

Here

$$\begin{aligned} \epsilon_p &= \frac{p^2}{2m} + \int dp' \{V(0) - \frac{1}{2}V(|p - p'|)\} f_{p'}(r, t) \\ h_p &= -\frac{1}{2}(\gamma B - \int dp' V(p - p') \bar{\sigma}_{p'}(r, t), \end{aligned}$$

$V$  is the inter-particle potential,  $B$  an external magnetic field, and  $\gamma$  is the gyromagnetic ratio.

In **spintronics** for semiconductor hetero-structures, related linear spinor Boltzmann equations

$$\frac{\partial}{\partial t} \rho + v \cdot \nabla_x \rho + E \cdot \nabla_v \rho = Q(\rho) + Q_{SO}(\rho) + Q_{SF}(\rho). \quad (2.7)$$

are considered. Here  $E$  is an electric field and  $Q$  is the collision operator for collisions without spin-reversal, in the linear BGK approximation

$$\int_{\mathbb{R}^3} \alpha(v, v') (M(v) \rho(v') - M(v') \rho(v)) dv',$$

$M$  denoting a normalized Maxwellian. The spin-orbit coupling generates an effective field  $\Omega$  making the spins precess. The corresponding spin-orbit interaction term  $Q_{SO}(\rho)$  is given by  $\frac{i}{2}[\Omega \cdot \sigma, \rho]$ . Finally  $Q_{SF}(\rho)$  is a spin-flip collision operator, in relaxation time approximation given by

$$Q_{SF}(\rho) = \frac{\text{tr} \rho I_2 - 2\rho}{\tau_{sf}},$$

with  $\tau_{sf} > 0$  the spin relaxation time.

Mathematical properties such as existence, uniqueness, and asymptotic behaviour, have been studied in particular by the French group around Ben Abdallah, his coworkers and students (see [BH] and references therein).

### 3 a) Low temperature bosons and a condensate.

For a Bose fluid and for temperatures sufficiently low for the mass of the condensate noticeably to exceed the mass of the (quasi-)particles, only interactions between thermally excited (quasi-)particles and a condensate are of practical importance. We shall now consider one such situation allowing transfer of atoms between the components, so that in particular no mass conservation should be expected for an individual component. The model (cf [KK], [K], [E] and for validation [S1]) is based on the Beliaev-Popov approximation of the Bogoliubov Green-function description which ignores off-diagonal correlations, and the Thomas-Fermi approximation which neglects a quantum pressure term.

Our simplified two-component model consists of a kinetic equation for the distribution function of a gas of bosonic **(quasi-)particles interacting with a Bose condensate**, which in turn is described by a Gross-Pitaevskii equation (cf [PS]).

In the local rest frame, the kinetic equation for the excitations is

$$\frac{\partial f}{\partial t} + \frac{1}{\epsilon} \nabla_p (E_p + v_s p) \cdot \nabla_x f - \frac{1}{\epsilon} \nabla_x (E_p + v_s p) \cdot \nabla_p f = \frac{1}{\epsilon^2} Q(f, n_c). \quad (3.1)$$

Here  $v_s$  is the superfluid velocity,  $E_p$  the Bogoliubov excitation energy, and  $\epsilon$  a small parameter. The collision term becomes

$$Q(f, n_c)(p) = n_c \int |A|^2 \delta(p_1 - p_2 - p_3) \delta(E_1 - E_2 - E_3) [\delta(p - p_1) - \delta(p - p_2) - \delta(p - p_3)] ((1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)) dp_1 dp_2 dp_3. \quad (3.2)$$

We notice that the *domain of integration* is only 2d and not 5d as for classical Boltzmann. The transition probability kernel  $|A|^2$  can be explicitly computed by the Bogoliubov approximation scheme.

$$A := (u_3 - v_3)(u_1 u_2 + v_1 v_2) + (u_2 - v_2)(u_1 u_3 + v_1 v_3) - (u_1 - v_1)(u_2 v_3 + v_2 u_3).$$

Here the Bose coherence factors  $u$  and  $v$  are

$$u_p^2 = \frac{\tilde{\epsilon}_p + E_p}{2E_p}, \quad u_p^2 - v_p^2 = 1,$$

with  $\tilde{\epsilon}_p = \frac{p^2}{2m} + gn_c$ ,  $n_c$  the non-equilibrium density of the atoms in the condensate,  $m$  the atomic mass, and  $g$  a scattering length defined later.

The collision operator  $Q(f, n_c)$  can be formally obtained (cf [ST], [EMV], [No]) from the Nordheim-Uehling-Uhlenbeck collision operator

$$\begin{aligned} \tilde{Q}_{NUU}(f)(p) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} B \delta(p + p_* - (p' + p'_*)) \delta(E(p) + E(p_*) - (E(p') + E(p'_*))) \\ (f' f'_*(1 + f)(1 + f_*) - f f_*(1 + f')(1 + f'_*)) dp_* dp' dp'_*. \end{aligned}$$

Namely, assume that a condensate appears below the Bose-Einstein condensation temperature  $T_c$ . That splits the quantum gas distribution function into a condensate part  $n_c \delta_{p=0}$  and an  $L^1$ -density part  $f(t, x, p)$ , and gives

$$\tilde{Q}_{NUU}(f + n_c \delta_{p=0}) = \tilde{Q}_{NUU}(f) + Q(f, n_c) + b n_c^2 + c n_c^3 + d n_c \delta_{p=0},$$

where a simple computation shows that  $b = c = 0$ .

At the low temperatures we have in mind here, the number of excited (quasi-)particles are considered to be small, when they are sufficiently excited for pair collisions to be important. Also the time-scale to reach equilibrium for such collisions, is considered to be short compared to the one for  $Q$ , and so the  $\tilde{Q}_{NUU}$  collision term is neglected relative to the collision operator  $Q$ .

The usual **Gross-Pitaevskii (GP) equation** for the wave function  $\psi$  (the order parameter) associated with a Bose condensate is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta_x \psi + (U_{ext} + g|\psi|^2)\psi,$$

i.e. a Schrödinger equation complemented by a non-linear term accounting for two-body interactions.  $U_{ext}$  is an external potential. For simplicity, we here do not include the strongly space-inhomogeneous trapping potential, otherwise an essential ingredient in experiments on laser-trapped Bose gases. Modulo a numerical factor,  $g$  is the  $s$ -scattering length of the two-body interaction potential. In the present context the GP equation is further generalized by letting the condensate move in a self-consistent (Hartree-Fock) mean field  $2g\tilde{n} = 2g \int f(p)dp$  produced by the thermally excited atoms, together with a dissipative coupling term associated with the collisions. This gives

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{\epsilon^2} \Delta_x \psi + (U_{ext} + g|\psi|^2 - i\frac{1}{2m\epsilon^2} \int C(f, n_c)(p)dp)\psi, \quad (3.3)$$

with  $C(f, n_c) = \frac{1}{n_c} Q(f, n_c)$  and  $U_{ext}$  in our case equal to a constant plus a term of order  $\epsilon$ . It is often useful to split the equation for  $\psi = \sqrt{n_c} e^{i\theta}$  into phase and amplitude variables (polar representation or the Madelung transform) and remove some physically negligible term, leading to

$$\begin{aligned} \epsilon^2 \frac{\partial n_c}{\partial t} + \nabla_x \cdot (n_c v_s) &= - \int Q(f, n_c) dp \\ \frac{\partial \theta}{\partial t} &= -\mu_c - \frac{mv_s^2}{2\epsilon^2}, \end{aligned}$$

with  $\mu_c$  a local condensate chemical potential.

### 3 b) A space-homogeneous, isotropic case.

Consider the Cauchy problem for the two component model (3.1), (3.3) in a space-homogeneous isotropic situation and in the superfluid rest frame (condensate velocity  $v_s = \nabla_x \theta = 0$ ), i.e. the equations

$$\frac{\partial f}{\partial t} = Q(f, n_c), \quad (3.4)$$

$$\frac{dn_c}{dt} = - \int Q(f, n_c) dp, \quad (3.5)$$

with initial values

$$f(p, 0) = f_i(|p|), \quad n_c(0) = n_{ci}. \quad (3.6)$$

Here  $f(p, t)$  is the density of the quasi-particles,  $n_c(t)$  the mass of the condensate, and the collision operator  $Q$  is given by (3.2).



There are two physically important regimes (cf [E]). One is the very low temperature situation with all  $|p_i| \ll p_0$  ( $T \leq 0.4T_c$  in the set-up of [E]), i.e. where physically all quasi-particle momenta are much smaller than the characteristic momentum  $p_0 = \sqrt{2mgn_c}$  for the crossover between the linear and quadratic parts of the Bogoliubov excitation energy of the quasi-particles;

$$E(p) := \sqrt{\frac{p^4}{4m^2} + \frac{gn_c}{m}p^2} \approx c|p|(1 + \frac{p^2}{8gmnc}) = c|p|(1 + \frac{p^2}{4p_0^2}). \quad (3.7)$$

Here  $c := \sqrt{\frac{gn_c}{m}}$  the speed of Bogoliubov sound. Setting  $m = \frac{1}{\sqrt{2}}$  gives  $p_0 = c$ . In applications with  $|p| \ll p_0$ , the right hand side of (3.7) is usually taken as the value of  $E(p)$ .

The Bose coherence factors can then be taken as

$$u_p = \sqrt{\frac{gn_c}{2E(p)}} + \frac{1}{2}\sqrt{\frac{E(p)}{2gn_c}}, \quad v_p = \sqrt{\frac{gn_c}{2E(p)}} - \frac{1}{2}\sqrt{\frac{E(p)}{2gn_c}}, \quad u_p^2 - v_p^2 = 1,$$

which gives

$$A = \frac{1}{2^{\frac{5}{2}}} \frac{\sqrt{E(p_*)E(p')E(p'_*)}}{(gn_c)^{\frac{3}{2}}} + \sqrt{\frac{gn_c}{2}} \left( \sqrt{\frac{E(p'_*)}{E(p_*)E(p')}} + \sqrt{\frac{E(p')}{E(p_*)E(p'_*)}} - \sqrt{\frac{E(p_*)}{E(p')E(p'_*)}} \right).$$

And so recalling that  $E(p_*) = E(p') + E(p'_*)$ , we obtain

$$A = \frac{1}{2^{\frac{5}{2}}} \frac{\sqrt{E(p_*)E(p')E(p'_*)}}{(gn_c)^{\frac{3}{2}}}.$$

With this  $A$ , the collision operator becomes

$$Q(f, n_c)(p) = \int \chi \frac{E(p_1)E(p_2)E(p_3)}{g^3n_c^2} \delta(p_1 - p_2 - p_3) \delta(E_1 - E_2 - E_3) [\delta(p - p_1) - \delta(p - p_2) - \delta(p - p_3)] ((1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)) dp_1 dp_2 dp_3, \quad (3.8)$$

where  $\chi$  denotes the truncation for  $|p_i| \leq \lambda$ ,  $1 \leq i \leq 3$  for a given  $\lambda > 0$ .

The opposite limit where all momenta  $|p_i| \gg p_0$  has the dominant excitation of (Hartree-Fock) single particle type (in [E] for moderately low temperatures around  $0.7T_c$ ). Expanding the square root definition of  $E$  in (3.7), we may approximate  $E_p$  by  $\frac{p^2}{2m} + gn_c$  leading to a collision operator of the type

$$Q(f, n_c) = kn_c \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \chi \delta(p_1 - p_2 - p_3) \delta(E_1 - E_2 - E_3) [\delta(p - p_1) - \delta(p - p_2) - \delta(p - p_3)] ((1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)) dp_1 dp_2 dp_3 \quad (3.9)$$

for the 'partial local equilibrium regime'. Here  $\chi$  is the characteristic function of the set of  $(p, p_1, p_2, p_3)$  with  $|p|, |p_1|, |p_2|, |p_3| \geq \alpha$  for a given positive constant  $\alpha$ .

In the general case, the collision operator is

$$Q(f, n_c)(p) = n_c \int |A|^2 \delta(p_1 - p_2 - p_3) \delta(E_1 - E_2 - E_3) [\delta(p - p_1) - \delta(p - p_2) - \delta(p - p_3)] ((1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)) dp_1 dp_2 dp_3, \quad (3.10)$$

with the excitation energy  $E$  defined by

$$E(p) = |p| \sqrt{\frac{p^2}{4m^2} + \frac{gn_c}{m}}.$$

The kernel  $|A|^2$  is bounded by a multiple of

$$|\bar{A}|^2 := \left( \frac{|p_1|}{\sqrt{n_c}} \wedge 1 \right) \left( \frac{|p_2|}{\sqrt{n_c}} \wedge 1 \right) \left( \frac{|p_3|}{\sqrt{n_c}} \wedge 1 \right),$$

in the physically interesting cases where asymptotically all  $|p_i| \ll p_0$ , all  $|p_i| \gg p_0$ , or one  $|p_i| \ll p_0$  and the others  $\gg p_0$ . These three cases are relevant for very low respectively moderately low temperatures compared to  $T_c$ , and (the third case) for the collision of low energy phonons with high energy excitations (atoms). The asymptotic situation of two  $|p_i| \ll p_0$  and one  $p_i \gg p_0$  (with unbounded  $A$ ) is excluded by the energy condition.

Using  $|\bar{A}|^2$  as the kernel in the collision operator, we have proved the following result.

**Theorem 3.1** [AN1] *Let  $n_{ci} > 0$  and  $f_i(p) = f_i(|p|) \in L^1$  be given with  $f_i$  nonnegative and  $f_i(p)|p|^{2+\gamma} \in L^1$  for some  $\gamma > 0$ . For the collision operator (3.10) with the transition probability kernel  $|\bar{A}|^2$ , there exists a nonnegative solution  $(f, n_c) \in C^1([0, \infty); L_+^1) \times C^1([0, \infty))$  to the initial value problem (3.1-3) in the space-homogeneous, isotropic case. The condensate density  $n_c$  is locally bounded away from zero for  $t > 0$ . The excitation density  $f$  has energy locally bounded in time. Total mass  $M_0 = n_{ci} + \int f_i(p) dp$  is conserved, and the moment  $\int |p|^{2+\gamma} f dp$  is locally bounded in time. In the moderately low temperature case a total energy quantity is conserved.*

The proof is via approximations controlled by a priori estimates and fixed point techniques. An important still open problem is the question of time-asymptotics. The addition of a NUU collision term would (using methods we are aware of) considerably weaken the type of solutions that can be obtained.

A. Heintz has developed numerical methods for the above evolution problem in connection with an exam project [BDJ]. The recent paper [S2] considers the spatially homogeneous and isotropic kinetic regime of weakly interacting bosons with s-wave scattering. It has a focus on post-nucleation self-similar solutions. Another recent paper, [EPV], studies linearized space homogeneous kinetic problems in settings related to the ones discussed here, and with a focus on large time behaviour.

### 3 c) A space-dependent, close to equilibrium case.

In equilibrium, the right hand side of the kinetic equation (3.1), i.e. the collision term, vanishes. Multiplying with  $\frac{\log f}{1+\log f}$  and integrating in  $p$ , we obtain

$$\frac{f_*}{(1+f_*)} = \frac{f'}{(1+f')} \frac{f'_*}{(1+f'_*)}. \quad (3.11)$$

As usual - and ultimately because of a Cauchy equation - equation (3.11) implies that  $f = \frac{M}{(1-M)}$  with  $M$  a Maxwellian. We assume that the Maxwellian has zero bulk velocity,  $M = \exp(-\beta \frac{p^2}{2m} + \delta)$ . With  $p_0 = \sqrt{2mgn_c}$ , we restrict to the 'intermediate temperature region'  $|p| \gg p_0$  in the superfluid rest frame. Denoting the mass density of the superfluid by  $n_0$ , and using the approximation

$\frac{p^2}{2m} + gn_0$  for the energy  $E$  of (3.7), equation (3.11) holds for  $\delta = -\frac{\beta}{2m}gn_0$ . Hence for  $m = \frac{1}{2}$ , the particular Maxwellian  $\mathcal{M}_0 = \exp(-\beta(p^2 + gn_0))$  gives an equilibrium for the collision operator  $Q$ . Take for simplicity  $g = \beta = 1$ , which fixes the equilibrium limit as  $(P_0, n_0)$  with  $P_0 = \frac{1}{(\exp(p^2 + n_0) - 1)}$ , the Planckian corresponding to  $\mathcal{M}_0$ . Require that the ingoing boundary values for the kinetic part are equal to this limit. Take the initial values  $(f_0, n_0)$  as  $\epsilon^3$ -small deviations from this equilibrium which retain the total mass  $M_0$ .

This space-dependent, close to equilibrium evolutionary problem poses interesting questions about existence (work in progress with A. Nouri) and the long time behaviour, including whether the solution converges to equilibrium when time tends to infinity. A collision term of NUU type can easily be included. In contrast to the NUU case of [R], here the splitting of the linearized collision operator into collision frequency plus a compact operator, requires a (physically acceptable) *cut-off* for very large velocities. Also the Milne problem is different from the classical case, e.g. in not having a constant mass flow, and again requires a truncation for large  $p$ 's.

## 4 a) A kinetic anyon equation.

The quantum Boltzmann Haldane equation [BH] is a kinetic equation of Boltzmann type for confined quasi-particles with *Haldane statistics* [H]. This exclusion statistics interpolates between the Fermi and Bose quantum behaviours. In quantum statistical mechanics, the number of quantum states of  $N$  identical particles occupying  $G$  states is given by

$$\frac{(G + N - 1)!}{N!(G - 1)!} \quad \text{and} \quad \frac{G!}{N!(G - N)!}$$

in the boson resp. fermion cases. The interpolated number of quantum states for the fractional exclusion of Haldane and Wu is

$$\frac{(G + (N - 1)(1 - \alpha))!}{N!(G - \alpha N - (1 - \alpha))!} \quad 0 < \alpha < 1. \quad (4.1)$$

When applied to the fractional quantum Hall effect, the Haldane statistics coincides with the two-dimensional anyon definition (in terms of the braiding of particle trajectories).

Haldane statistics may also be realized for neutral fermionic atoms at ultra-low temperature in three dimensions.

Elastic pair collisions in a Boltzmann type collision operator, are expected to preserve mass, linear momentum, and energy. That holds for the collision operator  $Q$  of Haldane statistics, first introduced in [BBM],

$$Q(f) = \int_{\mathbb{R}^d \times S^{d-1}} B(v - v_*, \omega) \times [f' f'_* F(f) F(f_*) - f f_* F(f') F(f'_*)] dv_* d\omega. \quad (4.2)$$

Here  $d\omega$  corresponds to the Lebesgue probability measure on the sphere.

The collision kernel  $B$  in the variables  $(z, \omega) \in \mathbb{R}^d \times S^{d-1}$  is positive, locally integrable, and only depends on  $|z|$  and  $|(z, \omega)|$ . The filling factor  $F$  is given by  $F(f) = (1 - \alpha f)^\alpha (1 + (1 - \alpha)f)^{1-\alpha}$ ,  $0 < \alpha < 1$ . The factor  $(1 - \alpha f)$  keeps the value of  $f$  between 0 and  $\frac{1}{\alpha}$ .  $F$  is convex with maximum value one at  $f = 0$  for  $\alpha \geq \frac{1}{2}$ , and maximum value  $(\frac{1}{\alpha} - 1)^{1-2\alpha} > 1$  at  $f = \frac{1-2\alpha}{\alpha(1-\alpha)}$  for  $\alpha < \frac{1}{2}$ .

With this filling factor, the collision operator vanishes identically for the Haldane equilibrium distribution functions as obtained in [W] under (4.1), but for no other functions. For the limiting cases, representing boson statistics ( $\alpha = 0$ ) and fermion statistics ( $\alpha = 1$ ) the quartic terms in the collision integral are canceled, and ideas from Lions' compactness result for the classical gain term are applicable. This allows approaches related to the weak  $L^1$  analysis for the quadratic Boltzmann equation.

But for  $0 < \alpha < 1$  there is *no cancellations* in the collision term. Moreover, the Lipschitz continuity of the collision term in the Fermi-Dirac case, is now replaced by a weaker Hölder continuity. For the space homogeneous case, I have developed an approach based on strong  $L^1$ -compactness.

Consider the initial value problem for the Boltzmann equation with Haldane statistics in the space-homogeneous case with velocities in  $\mathbb{R}^d$ ,  $d \geq 2$ ,

$$\frac{df}{dt} = Q(f). \quad (4.3)$$

Because of the filling factor  $F$ , the range for the initial value  $f_0$  should belong to  $[0, \frac{1}{\alpha}]$ , which is then formally preserved by the equation. The general BH equation ( $0 < \alpha < 1$ ) retains important properties from the Fermi-Dirac case, but it has so far not been validated from basic quantum theory. In particular the choice of kernel for the collision operator has to be studied in other ways. The choice by the physicists has been to use classical kernels such as those for hard forces,  $B(z, \omega) = |z|^\beta b(\frac{(z, \omega)}{|z|})$  with  $0 < \beta \leq 1$  and Grad angular cut-off, which I will now discuss. That also agrees with the better understood limiting cases  $\alpha = 0, 1$ . We assume that  $0 < b \leq c|z|^\beta |\sin \theta \cos \theta|^{d-1}$ , with an initial moment  $(1 + |v|^s)f_0$  in  $L^\infty$  for  $s \geq d - 1 + \beta$ .

**Theorem 4.1** [A3] *Consider the space-homogeneous equation (4.3) for hard force kernels with*

$$0 < B(z, \omega) \leq C|z|^\beta |\sin \theta \cos \theta|^{d-1}, \quad (4.4)$$

*where  $0 < \beta \leq 1$ ,  $d > 2$ , and  $0 < \beta < 1$ ,  $d = 2$ , and let the initial value  $0 < f_0 \in L^1$  have finite energy. If  $0 < f_0 \leq \frac{1}{\alpha}$  and  $\text{esssup}(1 + |v|^s)f_0 < \infty$  for  $s = d - 1 + \beta$ , then the initial value problem for (4.3) has a solution in the space of functions continuous from  $t \geq 0$  into  $L^1 \cap L^\infty$ , which conserves mass and energy, and for  $t_0 > 0$  given, has  $\text{esssup}_{v, t \leq t_0} |v|^{s'} f(t, v)$  bounded, where  $s' = \min(s, \frac{2\beta(d+1)+2}{d})$ .*

Idea of proof. This initial value problem is first considered for a family of approximations with bounded support for the kernel  $B$ , when  $0 < f_0 \leq \text{esssup} f_0 < \frac{1}{\alpha}$ . Starting from approximations with Lipschitz continuous filling factor, the corresponding solutions are shown to stay away uniformly from  $\frac{1}{\alpha}$ , the upper bound for the range. Uniform Lipschitz continuity follows for the approximating operators and leads to well posedness for the limiting problem. Uniform  $L^\infty$  moment bounds for the approximate solutions hold, using an approach from the classical Boltzmann case [A1]. Based on those preliminary results the global existence result for hard forces of Theorem 4.1, can be proved by strong compactness arguments. Mass and first moments are conserved and energy is bounded by its initial value. That bound on energy in turn implies energy conservation using the arguments for energy conservation from [MW] or [Lu2]. ■

**Remarks.**

i) The proof of the theorem implies the following stability. Given a sequence of initial values  $(f_{0n})_{n \in \mathbb{N}}$  with

$$\sup_n \operatorname{esssup}_v f_{0n}(v) < \frac{1}{\alpha},$$

and converging in  $L^1$  to  $f_0$ , then there is a subsequence of the solutions converging in  $L^1$  to a solution with initial value  $f_0$ .

ii) The question of long time behaviour is open.

iii) The space-dependent case is of interest also in the perspective of the general existence theory for Boltzmann type equations.

## 5 Differences between quantum and classical Boltzmann theory.

The examples discussed above, were selected to illustrate how the situation in quantum kinetic theory in various ways markedly differ from classical kinetic theory. The selection could be expanded to other types of low temperature kinetic evolutions of collision type, such as collisions involving five quasi-particles, two colliding ones giving rise to three or conversely, cf [K]. Other examples are Grassmann algebra valued gas densities, i.e. completely anti-commuting densities, e.g. for modelling the Pauli exclusion effect, cf [PR], and pair collision terms taking account of the finite duration of a collision, cf [ES].

i) Qualitatively different collision operators.

The type of collision operator varies qualitatively much more in the quantum regime than in the classical one, like (1.1), (1.2), (2.1), (2.7), (3.2), (4.2), and the examples mentioned from [K], [PR], and [ES]. The quantum influence sometimes appears as entirely new phenomena such as the collision-driven spin waves in the fermionic spin matrix Boltzmann equation. The thermal de Broglie wave length may become much larger than the typical inter-particle distance, an aspect totally absent from the classical theory. The quantum influence leads to different filling factors which, as for the NUU equation, can stabilize as well as destabilize the collision effects.

ii) Fewer collisions.

There are fewer collisions close to absolute zero, where permitted energy levels usually are discrete. In the example of boson excitations plus a condensate, an excitation may interact only with the modes of the zero-point motion that do not give away energy to it. Also, in that example the domain of integration has lower dimension than the classical case.

iii) Typical energy range.

A quantum situation is usually not scale invariant as in the classical case, but may have a typical energy range (Section 3b). A particular type of quasi-particles/excitations exists in a particular energy interval, and extending the energy outside this interval, may introduce collision types not observed in the experiments, and physically irrelevant for the modelling at hand. In this way bounded velocity domains may be both physically correct and mathematically adequate in quantum situations, like the two component, space-dependent boson example. Similarly, models and solutions with respect to a bounded interval of time, e.g. depending on relaxation effects, may be

relevant in particular quantum situations, such as the exciton-polariton case of [DHY].

iv) Parameter range and kernel family.

The collision terms and energy expressions discussed, were defined from a physical context. Changing a parameter mildly may completely change the mathematical aspects of the problem, as in the anyon example for  $\alpha$  around 0 and 1. Another example where very small changes in a parameter completely changes the physical situation, is the the extremely narrow temperature-pressure domain for the phenomenologically very rich A-phase in  $^3\text{He}$ . The question of parameter range and kernel family, is in general more delicate for quantum than for classical kinetic theory.

v) Questions from classical kinetic theory.

For each type of quantum kinetic collision term, the multitude of questions from classical kinetic theory can be posed and studied. New boundary-condition dependent phenomena may appear, an example being a low-temperature gas of excitations in a condensate between rotating cylinders having a vacuum friction type of radiation as in the Zeldovich-Starobinsky effect, cf [V].

vi) New insights.

The quantum problems may - in comparison with their classical counterparts - require new approaches or additional ideas for their solution, like the Milne problem for the space-dependent boson system.

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